SPECIAL EMBEDDINGS OF FINITE-DIMENSIONAL COMPACTA IN EUCLIDEAN SPACES

SEMEON BOGATYI AND VESKO VALOV

Abstract. If g is a map from a space X into $\mathbb{R}^m$ and $z \notin g(X)$, let $P_{2,1,m}(g,z)$ be the set of all lines $\Pi^1 \subset \mathbb{R}^m$ containing z such that $|g^{-1}(\Pi^1)| \geq 2$. We prove that for any n-dimensional metric compactum X the functions $g\colon X \to \mathbb{R}^m$, where $m \geq 2n+1$, with $\dim P_{2,1,m}(g,z) \leq 0$ for all $z \notin g(X)$ form a dense G_δ-subset of the function space $C(X,\mathbb{R}^m)$. A parametric version of the above theorem is also provided.

1. Introduction

In this paper we assume that all topological spaces are metrizable and all single-valued maps are continuous.

Everywhere below by $M_{m,d}$ we denote the space of all d-dimensional planes Π^d (br., d-planes) in $\mathbb{R}^m$. If g is a map from a space X into $\mathbb{R}^m$, q is an integer and $z \notin g(X)$, let $P_{q,d,m}(g,z) = \{\Pi^d \in M_{m,d} : |g^{-1}(\Pi^d)| \geq q \text{ and } z \in \Pi^d\}$. There is a metric topology on $M_{m,d}$, see [6], and we consider $P_{q,d,m}(g,z)$ as a subspace of $M_{m,d}$ with this topology.

One of the results from authors' paper [4] states that if X is a metric compactum of dimension n and $m \geq 2n+1$, then the function space $C(X,\mathbb{R}^m)$ contains a dense G_δ-subset of maps g such that the set $\{\Pi^1 \in M_{m,1} : |g^{-1}(\Pi^d)| \geq 2\}$ is at most $2n$-dimensional. The next theorem provides more information concerning the above result:

Theorem 1.1. *Let X be a metric compactum of dimension $\leq n$ and $m \geq 2n+1$. Then the maps $g\colon X \to \mathbb{R}^m$ such that $\dim P_{2,1,m}(g,z) \leq 0$ for all $z \notin g(X)$ form a dense G_δ-subset of $C(X,\mathbb{R}^m)$.*

Theorem 1.1 admits a parametric version.

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Theorem 1.2. *Let $f\colon X\to Y$ be a perfect n-dimensional map between metrizable spaces with $\dim Y=0$, and $m\geq 2n+1$. Then the maps $g\colon X\to\mathbb{R}^m$ such that $\dim P_{2,1,m}(g|f^{-1}(y),z)\leq 0$ for all restrictions $g|f^{-1}(y)$, $y\in Y$, and all $z\not\in g(f^{-1}(y)$ form a dense G_δ-subset of $C(X,\mathbb{R}^m)$ equipped with the source limitation topology.*

For any map $g\in C(X,\mathbb{R}^m)$ and $z\not\in g(X)$ we also consider the set $D_{2,1,m}(g,z)$ consisting of points $y=(y_1,y_2)\in(\mathbb{R}^m)^2$ such that y_1 and y_2 belong to a line $\Pi^1\subset\mathbb{R}^m$ with $z\in\Pi^1$, and there exist two different points $x_1,x_2\in X$ with $g(x_i)=y_i$, $i=1,2$.

Theorem 1.3 below follows from the proof of Theorem 1.2 by considering the sets $D_{2,1,m}(g,z)$ instead of $P_{2,1,m}(g,z)$.

Theorem 1.3. *Let X,Y, f and m satisfy the hypotheses of Theorem 1.2. Then the maps $g\colon X\to\mathbb{R}^m$ such that $\dim D_{2,1,m}(g|f^{-1}(y),z)\leq 0$ for all restrictions $g|f^{-1}(y)$, $y\in Y$, and all $z\not\in g(f^{-1}(y)$ form a dense G_δ-subset of $C(X,\mathbb{R}^m)$.*

Recall that for any metric space (M,ρ) the source limitation topology on $C(X,M)$ can be describe as follows: the neighborhood base at a given function $f\in C(X,M)$ consists of the sets $B_\rho(f,\epsilon)=\{g\in C(X,M):\rho(g,f)<\epsilon\}$, where $\epsilon: X\to(0,1]$ is any continuous positive functions on X. The symbol $\rho(f,g)<\epsilon$ means that $\rho(f(x),g(x))<\epsilon(x)$ for all $x\in X$. It is well know that for metrizable spaces X this topology doesn't depend on the metric ρ and it has the Baire property provided M is completely metrizable.

2. Preliminaries

We need some preliminary information before proving Theorem 1.1. Everywhere in this section we suppose that q,m,d are integers with $0\leq d\leq m$ and $q\geq 1$. Moreover, the Euclidean space $\mathbb{R}^m$ is equipped with the standard norm $||.||_m$. We also suppose that X is a metric compactum and $\Gamma=\{B_1,B_2,..,B_q\}$ is a disjoint family consisting of q closed subsets of X. For any $g\in C(X,\mathbb{R}^m)$ and $z\not\in g(X)$ we denote by $P_\Gamma(g,z)$ the set

$$\{\Pi^d\in M_{m,d}: g^{-1}(\Pi^d)\cap B_i\neq\varnothing \text{ for each } i=1,..,q \text{ and } z\in\Pi^d\}.$$

Now, consider the open subset $\mathcal{R}^m_X$ of $C(X,\mathbb{R}^m)\times\mathbb{R}^m$ consisting of all pairs (g,z) with $z\not\in g(X)$. Define the set-valued map

$$\Phi_\Gamma\colon\mathcal{R}^m_X\to M_{m,d},\ \Phi_\Gamma(g,z)=P_\Gamma(g,z).$$

Proposition 2.1. Φ_Γ *is an upper semi-continuous and closed-valued map.*

Proof. Suppose $(g_0, z_0) \in \mathcal{R}_X^m$. We need to show that for any open $W \subset M_{m,d}$ containing $\Phi_\Gamma(g_0, z_0)$ there are neighborhoods $O(g_0) \subset C(X, \mathbb{R}^m)$ and $O(z_0) \subset \mathbb{R}^m$ with $O(g_0) \times O(z_0) \subset \mathcal{R}_X^m$ and $\Phi_\Gamma(g, z) \subset W$ for all $(g, z) \in O(g_0) \times O(x_0)$. Assume this is not true. Then there exists a sequence $\{(g_k, z_k)\}_{k\geq 1} \in \mathcal{R}_X^m$ converging to (g_0, z_0) and $\Pi_k^d \in P_\Gamma(g_k, z_k)$ with $\Pi_k^d \not\in W$, $k \geq 1$. For any $i \leq q$ and $k \geq 1$ there exists a point $x_k^i \in B_i \cap g_k^{-1}(\Pi_k^d)$. Since $A = \bigcup_{i\leq q} g_0(B_i) \subset \mathbb{R}^m$ is compact, we take a closed ball K in $\mathbb{R}^m$ with center the origin containing A in its interior. Because every $\Pi^d \in P_\Gamma(g_0, z_0)$ intersects A, we can identify $P_\Gamma(g_0, z_0)$ with $\{\Pi^d \cap K : \Pi^d \in P_\Gamma(g_0, z_0)\}$ considered as a subspace of $\exp(K)$ (here $\exp(K)$ is the hyperspace of all compact subset of K equipped with the Vietoris topology).

Because $\{g_k\}_{k\geq 1}$ converges in $C(X, \mathbb{R}^m)$ to g_0, we can assume that K contains each set $\bigcup_{i\leq q} g_k(B_i)$, $k \geq 1$. Hence, $g_k(x_k^i) \in K \cap \Pi_k^d$ for all $i \leq q$ and $k \geq 1$. Therefore, passing to subsequences, we may suppose that there exist points $x_0^i \in B_i$ $i \leq q$, and a plane $\Pi_0^d \in M_{m,d}$ such that each sequence $\{x_k^i\}_{k\geq 1}$, $i = 1, 2, .., q$, converges to x_0^i and $\{\Pi_k^d \cap K\}_{k\geq 1}$ converges to $\Pi_0^d \cap K$. So, $\lim\{g_0(x_k^i)\}_{k\geq 1} = g_0(x_0^i)$, $i = 1, 2, .., q$. Then each $\{g_k(x_k^i)\}_{k\geq 1}$ also converges to $g_0(x_0^i)$. Consequently, $g_0(x_0^i) \in \Pi_0^d$ for all i. Moreover, since $z_k \in \Pi_k^d$ for all k, we also have $z_0 \in \Pi_0^d$. Hence, $\Pi_0^d \in P_\Gamma(g_0, z_0)$, i.e., $\Pi_0^d \in W$. On the other hand, W is open in $M_{m,d}$ and $\lim\{\Pi_k^d \cap K\}_{k\geq 1} = \Pi_0^d \cap K$ implies that $\{\Pi_k^d\}_{k\geq 1}$ converges to Π_0^d in $M_{m,d}$. This yields $\Pi_k^d \in W$ for almost all k, a contradiction.

The above arguments also show that $P_\Gamma(g, z)$ is closed in $M_{m,d}$ for all $(g, z) \in \mathcal{R}_X^m$. So, $\Phi_{\Gamma,m,d}$ is a closed-valued map. $\square$

Let X and the integers q, d, m be as above. We choose a countable family $\mathcal{B}$ of closed subsets of X such that the interiors of the elements of $\mathcal{B}$ form a base for the topology of X. Let also

$$\mathcal{R}_X^m(k) = \{(g, z) \in C(X, \mathbb{R}^m) \times \mathbb{R}^m : ||z||_m \leq k \text{ and } \rho_m(z, g(X)) \geq 1/k,$$

where ρ_m is the standard Euclidean metric on $\mathbb{R}^m$ and k an integer. If $\Gamma \subset \mathcal{B}$ is a disjoint family of q elements, for any integers k, s and $\epsilon > 0$ we consider the set $\mathcal{H}_\Gamma(k, s, \epsilon)$ of all maps $g \in C(X, \mathbb{R}^m)$ such that each $P_\Gamma(g, z)$, where $(g, z) \in \mathcal{R}_X^m(k)$, can be covered by an open in $M_{m,d}$ family $\omega(g, z)$ satisfying the following conditions:

(1) $\mathrm{mesh}(\omega(g, z)) < \epsilon$;
(2) the order of $\omega(g, z)$ is $\leq s$ (i.e., each point from $M_{m,d}$ is contained in at most $s + 1$ elements of $\omega(g, z)$).

Proposition 2.2. *Any $\mathcal{H}_\Gamma(k, s, \epsilon)$ is open in $C(X, \mathbb{R}^m)$.*

Proof. Assume $g_0 \in \mathcal{H}_\Gamma(k,s,\epsilon)$. For any $(g_0,z) \in \mathcal{R}^m_X(k)$ let $W(g_0,z) = \bigcup\{U : U \in \omega(g_0,z)\}$. Obviously, we have $(g_0,z) \in \mathcal{R}^m_X(k)$ if and only if z belongs to the compact set $B(g_0) = \{z \in \mathcal{R}^m : ||z||_m \leq k \text{ and } \rho_m(z,g_0(X)) \geq 1/k\}$. Hence, $P_\Gamma(g_0,z) \subset W(g_0,z)$ for every $z \in B(g_0)$. According to Proposition 2.1, for any such z there exists an open neighborhood $O(z) \subset \mathbb{R}^m\backslash g_0(X)$ such that $P_\Gamma(g_0,u) \subset W(g_0,z)$ for all $u \in O(z)$. Next, shrink each $O(z)$ to an open set $V(z)$ such that $z \in V(z) \subset \overline{V(z)} \subset O(z)$. Then $\{V(z) : z \in B(g_0)\}$ is an open cover of $B(g_0)$ and we choose a finite subcover $\{V(z_j) : j = 1,2,.,p\}$. Let η be the distance between $B(g_0)$ and $\mathbb{R}^m\backslash V$, where $V = \bigcup_{j=1}^{j=p} V(z_j)$, and $A(z) = \{j : z \in O(z_j)\}$, $z \in O = \bigcup_{j=1}^{j=p} O(z_j)$. Choosing smaller neighborhoods $V(z_j)$, if necessarily, we may assume that $\eta < 1/k$. According to the choice of $O(z_j)$, we have

$$(3). \qquad P_\Gamma(g_0,z) \subset W(g_0,z_j) \text{ for any } z \in O \text{ and } j \in A(z)$$

Claim 1. *Let $g \in O(g_0,\eta)$ and $\rho_m(z,g(X)) \geq 1/k$, where $O(g_0,\eta)$ consists of all $g \in C(X,\mathbb{R}^m)$ such that $\rho_m(g_0(x),g(x)) < \eta$ for all $x \in X$. Then $\rho_m(z,g_0(X)) \geq (1/k)-\eta$ and $z \in \overline{V} \subset O$.*

Indeed, both $\rho_m(z,g_0(X)) < (1/k)-\eta$ and $g \in O(g_0,\eta)$ imply the existence of $x \in X$ with $\rho_m(g_0(x),g(x)) < 1/k$ which contradicts $\rho_m(z,g(X)) \geq 1/k$. So, for every z satisfying the hypotheses of Claim 1, we have $\rho_m(z,g_0(X)) \geq (1/k)-\eta$. This yields $z \in \overline{V} \subset O$.

Each $W(g_0,z_j)$ is the union of an open family in $M_{m,d}$ of order $\leq s$ and mesh $< \epsilon$. Thus, according to Claim 1, it suffices to show the next claim.

Claim 2. *There exists a neighborhood $O(g_0) \subset O(g_0,\eta)$ of g_0 satisfying the following condition: for any $z \in \overline{V}$ with $\rho_m(z,g_0(X)) \geq (1/k)-\eta$ there exists $j \in A(z)$ such that $P_\Gamma(g,z) \subset W(g_0,z_j)$ whenever $g \in O(g_0)$.*

Suppose the conclusion of Claim 2 is not true. Then for every $p \geq 1$ there exists a map $g_p \in O(g_0,\eta)$ with $\rho_m(g_0(x),g_p(x)) < 1/p$ for all $x \in X$, a point $z_p \in \overline{V}$ with $\rho_m(z_p,g_0(X)) \geq (1/k)-\eta$, and planes

$$(4) \qquad \Pi^d_p \in P_\Gamma(g_p,z_p)\backslash\bigcup\{W(g_0,z_j) : j \in A(z_p)\}.$$

Passing to subsequences, we may assume that the sequence $\{z_p\}_{p\geq 1}$ converges to a point $z_0 \in \overline{V}$ and $\{\Pi^d_p\}_{p\geq 1}$ converges in $M_{m,d}$ to a d-plane Π^d_0. Obviously, $\rho_m(z_0,g_0(X)) \geq (1/k)-\eta$. Since $z_p \in \Pi^d_p$, we also have $z_0 \in \Pi^d_0$. As in the proof of Proposition 2.1, we can see that $g_0^{-1}(\Pi^d_0)$ meets each element of Γ. Consequently, $\Pi^d_0 \in P_\Gamma(g_0,z_0)$. So, by (3), $\Pi^d_0 \in \bigcap\{W(g_0,z_j) : j \in A(z_0)\}$. This implies that $\Pi^d_p \in \bigcap\{W(g_0,z_j) :$

$j \in A(z_0)\}$ for almost all p. On the other hand, since $\lim z_p = z_0$, there exists p_0 such that $A(z_0) \subset A(z_p)$ for all $p \geq p_0$. So, by (4), $\Pi_p^d \notin \bigcup\{W(g_0, z_j) : j \in A(z_0)\}$ when $p \geq p_0$, a contradiction. $\square$

Corollary 2.3. *All maps $g \in C(X, \mathbb{R}^m)$ such that $\dim P_{q,d,m}(g, z) \leq s$ for all $z \notin g(X)$ form a G_δ-subset $\mathcal{H}_X(q, d, m, s)$ of $C(X, \mathbb{R}^m)$.*

Proof. It easily seen that each $g \in C(X, \mathbb{R}^m)$ and $z \notin g(X)$ we have $P_{q,d,m}(g, z) = \bigcup\{P_\Gamma(g, z) : \Gamma \subset \mathcal{B}$ is disjoint and $|\Gamma| = q\}$. Moreover, since $P_\Gamma(g, z)$ are closed in $M_{m,d}$ (by Proposition 2.1), we have $\dim P_{q,d,m}(g, z) \leq s$ if and only if $\dim P_\Gamma(g, z) \leq s$ for all Γ. This implies that $\mathcal{H}_X(q, d, m, s)$ is the intersection of the sets $\mathcal{H}_\Gamma(k, s, 1/p)$, where $k, p \geq 1$ are integers and $\Gamma \subset \mathcal{B}$ is a disjoint family of q elements. $\square$

3. Proof of Theorems 1.1 and 1.2

Recall that a real number v is called algebraically dependent on the real numbers $u_1, .., u_k$ if v satisfies the equation $p_0(u) + p_1(u)v + ... + p_n(u)v^n = 0$, where $p_0(u), .., p_n(u)$ are polynomials in $u_1, .., u_k$ with rational coefficients, not all of them 0. A finite set of real numbers is *algebraically independent* if none of them depends algebraically on the others. The idea to use algebraically independent sets for proving general position theorems was originated by Roberts in [9]. This idea was also applied by Berkowitz and Roy in [3]. A proof of the Berkowitz-Roy main theorem from [3] was provided by Goodsell in [8, Theorem A.1] (see [5, Corollary 1.2] for a generalization of the Berkowitz-Roy theorem and [7] for another application of this theorem). Let us note that any finitely many points in an Euclidean space $\mathbb{R}^n$ whose set of coordinates is algebraically independent are in general position.

Proof of Theorem 1.1. We have to show that the set $\mathcal{H}_X(2, 1, m, 0)$ of all maps $g \in C(X, \mathbb{R}^m)$ such that $\dim P_{2,1,m}(g, z) \leq 0$ for all $z \notin g(X)$ is dense and G_δ in $C(X, \mathbb{R}^m)$. According to Corollary 2.3, this set is G_δ. So, it remains to show it is also dense in $C(X, \mathbb{R}^m)$. Fix a countable family $\mathcal{B}$ of closed subsets of X such that the interiors of its elements is a base for X. Since $\mathcal{H}_X(2, 1, m, 0)$ is the intersection of the open family

$$\{\mathcal{H}_\Gamma(k, 0, 1/p) : \Gamma \subset \mathcal{B} \text{ is disjoint with } |\Gamma| = 2 \text{ and } k, p \geq 1\}$$

(see the proof of Corollary 2.3), it suffices to show that each $\mathcal{H}_\Gamma(k, 0, \epsilon)$ is dense in $C(X, \mathbb{R}^m)$. Recall that $\mathcal{H}_\Gamma(k, 0, \epsilon)$ consists of all maps $g \in C(X, \mathbb{R}^m)$ such that $P_\Gamma(g, z)$ can be covered by a disjoint open in $M_{m,1}$ family ω with $\mathrm{mesh}(\omega) < \epsilon$ for every map g and every point $z \in \mathbb{R}^m$ satisfying the following conditions: $||z||_m \leq k$ and $\rho_m(z, g(X)) \geq 1/k$.

To prove that each $\mathcal{H}_\Gamma(k,0,\epsilon)$ is dense in $C(X,\mathbb{R}^m)$, observe that any map $g\in C(X,\mathbb{R}^m)$ can be approximated by maps $f=h\circ l$ with $l\colon X\to K$ and $h\colon K\to\mathbb{R}^m$, where K is a finite polyhedron of dimension $\le n$. Actually, K can be supposed to be a nerve of a finite open cover β of X. Moreover, if we choose β such that any its element meets at most one element of $\Gamma=\{B_1,B_2\}$, then we have $l(B_1)\cap l(B_2)=\varnothing$. Further, taking sufficiently small barycentric subdivision of K, we can find disjoint subpolyhedra K_i of K with $l(B_i)\subset K_i$, $i=1,2$. Obviously, for any $z\notin h(l(X))$ the set $P_\Gamma(h\circ l,z)$ is contained in $P_\Lambda(h,z)=\{\Pi^1\in M_{m,1}: h^{-1}(\Pi^1)\cap K_i\neq\varnothing, i=1,2 \text{ and } z\in\Pi^1\}$, where $\Lambda=\{K_1,K_2\}$. Therefore, the density of $\mathcal{H}_\Gamma(k,0,\epsilon)$ in $C(X,\mathbb{R}^m)$ is reduced to show that the maps $h\in C(K,\mathbb{R}^m)$ such that any $P_\Lambda(h,z)$, $z\notin h(K)$, admits a disjoint open cover in $M_{m,1}$ of mesh $<\epsilon$ form a dense subset of $C(K,\mathbb{R}^m)$. And this follows the next proposition.

Proposition 3.1. *Let K_i, $i=1,2$, be disjoint n-dimensional subpolyhedra of a finite polyhedron K. Then the maps $h\in C(K,\mathbb{R}^m)$ such that for any $z\notin h(K)$ the set $\{\Pi^1\in M_{m,1}: h^{-1}(\Pi^1)\cap K_i\neq\varnothing, i=1,2$ and $z\in\Pi^1\}$ is of dimension ≤ 0 form a dense subset of $C(K,\mathbb{R}^m)$.*

Proof. Let $h_0\in C(K,\mathbb{R}^m)$ and $\delta>0$. We take a subdivision of K such that $\mathrm{diam} h_0(\sigma)<\delta/2$ for all simplexes σ. Let $K^{(0)}=\{a_1,a_2,...,a_t\}$ be the vertexes of K and $v_j=h_0(a_j)$, $j=1,..,t$. Then, by [3], there are points $b_j\in\mathbb{R}^m$ such that the distance between v_j and b_j is $<\delta/2$ for each j and the coordinates of all b_j, $j=1,..,t$, form an algebraically independent set. Define a map $h\colon K\to\mathbb{R}^m$ by $h(a_j)=b_j$ and h is linear on every simplex of K. It is easily seen that h is δ-close to h_0. Without loss of generality, we may suppose that K_1 and K_2 are two n-dimensional simplexes. Then each $h(K_i)$ is also an n-dimensional simplex in $\mathbb{R}^m$ generating a plane $\Pi^n_i\in M_{m,n}$. Since the coordinates of the points $\{b_j: j=1,..,t\}$ form an algebraically independent set, the planes Π^n_1 and Π^n_2 are skew. Suppose $z\notin h(K)$. If $z\in\Pi^n_1$ or $z\in\Pi^n_2$, then there is no line $\Pi^1\subset\mathbb{R}^m$ which contains z and meets both $h(K_1)$ and $h(K_2)$. Suppose $z\notin\Pi^n_1\cup\Pi^n_2$. According to [4, Corollary 3.8], there exists at most one line $\Pi^1\subset\mathbb{R}^m$ containing z such that $\Pi^1\cap h(K_i)\neq\varnothing$, $i=1,2$. Hence, for any $z\notin h(K)$ the set $\{\Pi^1\in M_{m,1}: h^{-1}(\Pi^1)\cap K_i\neq\varnothing, i=1,2$ and $z\in\Pi^1\}$ is finite. □

Proof of Theorem 1.2. We fix a metric d generating the topology of X and for any $g\in C(X,\mathbb{R}^m)$, $y\in Y$, $\eta>0$ and $z\notin g(f^{-1}(y))$ let $P^\eta(g,y,z)$ be the set of all $\Pi^1\in M_{m,1}$ such that $z\in\Pi^1$ and there exist

two points $x^1, x^2 \in g^{-1}(\Pi^1) \cap f^{-1}(y)$ with $d(x^1, x^2) \geq \eta$. Obviously,

$$(5) \quad P_{2,1,m}(g|f^{-1}(y), z) = \bigcup_{k=1}^{\infty} \{P^{1/k}(g, y, z) \text{ for any } z \notin g(f^{-1}(y))\}.$$

Claim 3. *Each $P^\eta(g, y, z)$ is closed in $P_{2,1,m}(g|f^{-1}(y), z)$.*

The proof of Claim 3 follows the arguments from the proof of Proposition 2.1.

Now, for $k \geq 1$ and $y \in Y$ consider the set

$$B_g(y, k) = \{z \in \mathbb{R}^m : ||z||_m \leq k \text{ and } \rho_m(z, g(f^{-1}(y))) \geq 1/k\}.$$

Next, let $\mathcal{P}^\eta_\epsilon(y, k)$ be the set of all maps $g \in C(X, \mathbb{R}^m)$ such that for each $z \in B_g(y, k)$ the set $P^\eta(g, y, z)$ can be covered by a disjoint open in $M_{m,1}$ family of mesh $< \epsilon$. If $F \subset Y$, we consider the set $\mathcal{P}^\eta_\epsilon(F, k) = \bigcap_{y \in F} \mathcal{P}^\eta_\epsilon(y, k)$. Obviously the intersection of all $\mathcal{P}^\eta_{1/s}(Y, k)$, $s \geq 1$, is the set

$$\mathcal{P}^\eta(Y, k) = \{g \in C(X, \mathbb{R}^m) : \dim P^\eta(g, y, z) \leq 0, y \in Y, z \in B_g(y, k)\}.$$

It follows from (5) that the set $\bigcap_{k,s=1}^{\infty} \mathcal{P}^{1/s}(Y, k)$ coincides with the set

$$\mathcal{P} = \{g \in C(X, \mathbb{R}^m) : \dim P_{2,1,m}(g|f^{-1}(y), z) \leq 0, y \in Y, z \notin g(f^{-1}(y)).$$

So, in order to show that $\mathcal{P}$ is dense and G_δ in $C(X, \mathbb{R}^m)$, it suffices to show that each $\mathcal{P}^\eta_\epsilon(Y, k)$ is open and dense in $C(X, \mathbb{R}^m)$.

We are going first to show that any $\mathcal{P}^\eta_\epsilon(Y, k)$ is open in $C(X, \mathbb{R}^m)$. This can be done following the arguments from [4, Proposition 5.3] using the next lemma instead of [4, Lemma 5.2].

Lemma 3.2. *Let $g_0 \in \mathcal{P}^\eta_\epsilon(y_0, k)$ for some $y_0 \in Y$. Then there exists a neighborhood V of y_0 in Y and $\delta > 0$ such that $g \in \mathcal{P}^\eta_\epsilon(V, k)$ for all $g \in C(X, \mathbb{R}^m)$ such that the restrictions $g|f^{-1}(V)$ and $g_0|f^{-1}(V)$ are δ-close.*

Proof. Assume the conclusion of Lemma 3.2 doesn't hold and use the arguments from the proof of Propositions 2.1 and 2.2 to obtain a contradiction. □

The next proposition completes the proof of Theorem 1.2.

Proposition 3.3. *Any set $\mathcal{P}^\eta_\epsilon(Y, k)$ is dense in $C(X, \mathbb{R}^m)$ with respect to the source limitation topology.*

Proof. We modify the arguments from the proof of [4, Proposition 5.4]. Let $g \in C(X, \mathbb{R}^m)$ and $\delta \in C(X, (0,1])$. We are going to find $h \in \mathcal{P}_\epsilon^\eta(Y,k)$ such that $\rho(g(x), h(x)) < \delta(x)$ for all $x \in X$. By [1, Proposition 4], g can be supposed to be simplicially factorizable. This means that there exists a simplicial complex D and maps $g_D\colon X \to D$, $g^D\colon D \to M$ with $g = g^D \circ g_D$. Following the proof of [2, Proposition 3.4], we can find an open cover $\mathcal{U}$ of X, simplicial complexes N, L and maps $\alpha : X \to N$, $\beta : Y \to L$, $p\colon N \to L$, $\varphi\colon N \to \mathbb{R}^m$ and $\delta_1\colon N \to (0,1]$ satisfying the following conditions, where $h' = \varphi \circ \alpha$:

- α is an $\mathcal{U}$-map and for any $x_1, x_2 \in X$ with $d(x_1, x_2) \geq \eta$ we have $\alpha(x_1) \neq \alpha(x_2)$;
- $\beta \circ f = p \circ \alpha$;
- p is a perfect PL-map with $\dim p \leq n$ and $\dim L = 0$;
- h' is $(\delta/2)$-close to g;
- $\delta_1 \circ \alpha \leq \delta$.

So, we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "h'"] \arrow[dr, "\alpha"] \arrow[dd, "f"'] & & \mathbb{R}^m \\
& N \arrow[ur, "\varphi"] \arrow[dd, "p"] & \\
Y \arrow[dr, "\beta"'] & & \\
& L &
\end{tikzcd}$$

Since L is a 0-dimensional simplicial complex and p is a perfect PL-map, N is a discrete union of the finite complexes $K_l = p^{-1}(l)$, $l \in L$. Because $\dim p \leq n$, $\dim K_l \leq n$, $l \in L$. Applying Theorem 1.1 to each complex K_l, we can find a map $\varphi_1\colon N \to \mathbb{R}^m$ such that for any $l \in L$ and $z \notin \varphi_1(p^{-1}(l))$ we have $\dim P_{2,1,m}(\varphi_1|p^{-1}(l), z) \leq 0$ and $\varphi_1|p^{-1}(l)$ is θ_l-close to $\varphi|p^{-1}(l)$, where $\theta_l = \min\{\delta_1(u) : u \in p^{-1}(l)\}$. Moreover, the map $h = \varphi_1 \circ \alpha$ is δ-close to g. We claim that $h \in \mathcal{P}_\epsilon^\eta(Y,k)$. Indeed, let $y \in Y$ and $z \in B_h(y,k)$. If $\Pi^1 \in P^\eta(h,y,z)$, then there exist two points $x^i \in h^{-1}(\Pi^1) \cap f^{-1}(y)$, $i = 1,2$, with $d(x^1, x^2) \geq \eta$. According to the choice of the cover $\mathcal{U}$, we have $\alpha(x^1) \neq \alpha(x^2)$. Since $\varphi_1^{-1}(\Pi^1) \cap p^{-1}(\beta(y))$ contains the points $\alpha(x^i)$, $i = 1,2$, we obtain that $\Pi^1 \in P_{2,1,m}(\varphi_1|p^{-1}(\beta(y)), z)$. Thus, we established the inclusion $P^\eta(h,y,z) \subset P_{2,1,m}(\varphi_1|p^{-1}(\beta(y)), z)$ which implies $\dim P^\eta(h,y,z) \leq 0$ for every $y \in Y$ and $z \in B_h(y,k)$. Consequently, $h \in \mathcal{P}_\epsilon^\eta(Y,k)$. $\square$

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Faculty of Mechanics and Mathematics, Moscow State University, Vorobevy gory, Moscow, 119899 Russia

E-mail address: `bogatyi@inbox.ru`

Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, P.O. Box 5002, North Bay, ON, P1B 8L7, Canada

E-mail address: `veskov@nipissingu.ca`